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Number theory

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Problem Set #11

Exercise 1 p 71
Let A be an arbitrary ring, not necessarily an integral domain, let M be an A-module and S a multiplicatively closed subset of A such that $0 \notin S$. In $M \times S$ consider the equivalence relation

$$(m, s) \sim (m', s') \Leftrightarrow \exists s'' \in S \text{ such that } s''(s'm - sm') = 0$$

Show that the set M_S of equivalence classes $\overline{(m, s)}$ forms an A-module, and that $M \to M_S$, $a \mapsto \overline{(a, 1)}$, is a homomorphism. In particular, A_S is a ring. It is called the **localization** of A with respect to S.

Solution:
For convenience, we will write $\frac{m}{s} := \overline{(m, s)}$.
We want to prove that the set M_S *of the equivalence relation forms an* A*-module.*
For this we define two operation $+$ *and* $\cdot$ *on* M_S*, as for any* $m, n \in M$, $s, t \in S$ *and* $a \in A$,

$$\frac{m}{s} + \frac{n}{t} := \frac{tm + sn}{st}$$

and

$$a \cdot \frac{m}{s} := \frac{am}{s}$$

Since it is equivalence classes we have to check that it is well defined.
For this, let $m', n' \in M$, *and* $s', t' \in S$ *such that* $\frac{m}{s} = \frac{m'}{s'}$ *and* $\frac{n}{t} = \frac{n'}{t'}$, *that is there exists* $s'', t'' \in S$ *such that*

$$s''(s'm - sm') = 0 \text{ and } t''(t'n - ts') = 0$$

. *We need to prove that*

$$\frac{tm + sn}{st} = \frac{t'm' + s'n'}{s't'}$$

that is there exist $r \in S$ *such that*

$$r((tm + sn)s't' - (t'm' + s'n')st) = 0$$

Put $r = s''t''$, *then*

$$\begin{array}{rcl} r((tm + sn)s't' - (t'm' + s'n')st) & = & s''t''((tm + sn)s't' - (t'm' + s'n')st) \\ & = & tt's''(s'm - sm') - ss't''(t'n - ts') \\ & = & 0 - 0 = 0 \end{array}$$

Similarly, we prove that $\frac{am}{s} = \frac{am'}{s'}$.
To prove that f *is a homomorphism, we have to prove that*

1. *For any $m, n \in M$, $f(m+n) = f(m) + f(n)$, that is $\frac{m+n}{1} = \frac{m}{1} + \frac{n}{1}$, and this is also clear by the definition of the operation which make M_S into a A-module.*

2. *For any $m \in M$ and $a \in A$, $f(am) = af(m)$, that is $\frac{am}{1} = a \cdot \frac{m}{1}$ and this is also clear.*
   Moreover, A_S is a ring, with unit $\frac{1}{1}$, neutral element $\frac{0}{1}$ and the multiplication defined by
   $$\frac{m}{s} \cdot \frac{n}{t} = \frac{mn}{st}$$
   As before we can easily check that it is well defined.

Exercise 2 p 72
Show that, in the above situation, the prime ideals of A_S correspond $1-1$ to the prime ideals of A which are disjoint from S. If $\mathfrak{p} \subseteq A$ and $\mathfrak{p}_S \subseteq A_S$ correspond in this way, then $A_S/\mathfrak{p}_S$ is the localization of $A/\mathfrak{p}$ with respect to the image of S.
Solution:
The one-to-one correspondence is given by associating to a prime ideal Q of A_S the ideal $Q \cap A$ of A and associating to a prime ideal $\mathfrak{q}$ of A disjoint from S the ideal $\mathfrak{q}A_S$ of A_S.
It is well define since when Q is a prime ideal of A_S clearly $Q \cap A$ is a prime ideal of A and if $\mathfrak{q}$ is a prime ideal of A disjoint from S, then $\mathfrak{q}A_S$ is a prime ideal of A_S, indeed let $\frac{a}{s}$ and $\frac{a'}{s'}$ such that $\frac{aa'}{ss'} \in \mathfrak{q}A_S$, that is $\frac{aa'}{ss'} = \frac{q}{t}$ for some $q \in \mathfrak{q}$ and $t \in S$. So that, there is a $r \in S$ such that $r(aa't - ss'q) = 0$ and $rtaa' = rss'q \in \mathfrak{q}$ but since $rt \notin \mathfrak{q}$ and $\mathfrak{q}$ is a prime ideal then $aa' \in \mathfrak{q}$ and so as $\mathfrak{q}$ is a prime ideal again $a \in \mathfrak{q}$ or $a' \in \mathfrak{q}$ and $\mathfrak{q}A_S$ is a prime ideal.
Now, we need to prove $(Q \cap A)A_S = Q$ and $\mathfrak{q}A_S \cap A = \mathfrak{q}$. Clearly, $(Q \cap A)A_S \subseteq Q$. Let $q \in Q$, then $q = \frac{u}{s}$ with $u \in a$ and $s \in S$, then $qs = u \in Q \cap A$ and $q \in (Q \cap A)A_S$.
Clearly, $\mathfrak{q} \subseteq \mathfrak{q}A_S \cap A$. Now, $a \in \mathfrak{q}A_S \cap A$, then $a = \frac{q}{s}$ then $as = q \in \mathfrak{q}$ but $s \notin \mathfrak{q}$ and $\mathfrak{q}$ is a prime ideal. Then $a \in \mathfrak{q}$.
Now, we prove that $A_S/\mathfrak{p}_S$ is the localization $(A/\mathfrak{p})_{f(S)}$ where $f : A \to A/\mathfrak{p}$. We define the morphism,
$$\begin{array}{rccc} \phi: & A_S & \to & (A/\mathfrak{p})_{f(S)} \\ & \frac{a}{s} & \mapsto & \frac{f(a)}{f(s)} \end{array}$$
It is well defined since $\frac{a}{s} = \frac{a'}{s'}$, then there is a $t \in S$ such that $t(s'a - sa') = 0$ but then $f(t)(f(s')f(a) - f(s)f(a')) = 0$ so that $\frac{f(a)}{f(s)} = \frac{f(a')}{f(s')}$. We can prove that ϕ is a surjective homomorphism with kernel $\mathfrak{p}_S$.

Exercise 3 p 72
Let $f : M \to N$ be a homomorphism of A-modules. Then the following conditions are equivalent:

1. f is injective (surjective).
2. $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective (surjective) for every prime ideal $\mathfrak{p}$.
3. $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ is injective (surjective) for every maximal ideal $\mathfrak{m}$.

Solution:
First we prove the following result: *Let M be an A-module. Then TFAE:*

1. $M = 0$;
2. $M_{\mathfrak{p}} = 0$, *for all prime ideal* $\mathfrak{p}$;
3. $M_{\mathfrak{m}} = 0$, *for all maximal ideal* $\mathfrak{m}$;

Proof: 1. $\Rightarrow$ 2. $\Rightarrow$ 3. *is obvious, so it remains only to prove that* 3. $\Rightarrow$ 2. *Suppose that* 3. *hold and M is not the zero module. Hence, there is an $x \in M$, $x \neq 0$. Now,*

$$Ann(x) = \{a \in A | a \in A, ax = 0\} \subseteq A$$

and certainly $1 \notin Ann(x)$, then $Ann(x) \subseteq \mathfrak{m}$, for some maximal ideal $\mathfrak{m}$ of A. But, by 3., *we have $M_{\mathfrak{m}} = 0$. In particular, $x/1$ is zero in $M_{\mathfrak{m}}$. Hence, there is a $u \in A \backslash \mathfrak{m}$, such that $u(1 \cdot x - 1 \cdot 0) = ux$. So $u \in Ann(x) \subseteq \mathfrak{m}$, contradicting that $u \notin \mathfrak{m}$. Thus $M = 0$, so* 1. *holds and the result is proved.*
We prove now the injective case.
1. $\Rightarrow$ 2. *If f is injective then$f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective. Let $\frac{m}{s} \in ker(f_{\mathfrak{p}})$, then $f(\frac{m}{s}) = \frac{f(m)}{f(s)} = 0$, so that there is a $t \in 1 \backslash \mathfrak{p}$ such that $tf(m) = 0$, so that $f(tm) = 0$ and since f is surjective $tm = 0$ and $\frac{m}{s} = 0$. This prove the injectivity of f.*
2. $\Rightarrow$ 3. *is obvious.*
3. $\Rightarrow$ 1. *Suppose that* 3. *holds, and put $M' = ker(\phi)$. Then*

$$0 \longrightarrow M' \xrightarrow{\phi} M \longrightarrow N$$

is exact where the second mapping is inclusion. For each maximal ideal $\mathfrak{m}$

$$0 \longrightarrow M'_{\mathfrak{m}} \xrightarrow{\phi} M_{\mathfrak{m}} \longrightarrow N_{\mathfrak{m}}$$

is exact, so that

$$M'_{\mathfrak{m}} = ker(\phi_{\mathfrak{m}}) = 0$$

by 3., *since $\phi_{\mathfrak{m}} = 0$ is injective. Using the result, staten at the beginning we get that $M' = 0$ which prove that ϕ is injective.*
The surjectivity is obtained similarly reversing the arrows in the previous argument and using the image instead of the kernel.

Exercise 5 p 72
Let $f : A \to B$ be a homomorphism of rings and S a multiplicatively close subset of A such that $f(S) \subseteq B^*$. Then f induces a homomorphism $g : A_S \to B$.
Solution:
We can define the morphism g as follow:

$$\begin{array}{cccc} g: & A_S & \to & B \\ & \frac{a}{s} & \mapsto & \frac{f(a)}{f(s)} \end{array}$$

If is well defined since $f(S) \subseteq B^*$ and if $\frac{a}{s} = \frac{a'}{s'}$ that is $u \in S$ such that $u(s'a - sa') = 0$, then $f(u(s'a - sa')) = f(0) = 0$, and since f is a morphism, $f(u)(f(s')f(a) -$

$f(s)f(a')) = 0$ that is $\frac{f(a)}{f(s)} = \frac{f(a')}{f(s')}$.
Now, g is a homomorphism since for any $\frac{a}{s}, \frac{a'}{s'}$, we have

$$g(\frac{a}{s} + \frac{a'}{s'}) = g(\frac{as' + a's}{ss'}) = \frac{f(a)f(s') + f(a')f(s)}{f(s)f(s')} = \frac{f(a)}{f(s)} + \frac{f(a')}{f(s')} = g(\frac{a}{s}) + g(\frac{a'}{s'})$$

$$g(\frac{a}{s}\frac{a'}{s'}) = g(\frac{a}{s})g(\frac{a'}{s'})$$

$$g(\frac{1}{1}) = \frac{1}{1}$$

Note that g is the unique morphism making the following diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "f"'] & A_S \arrow[d, "g"] \\ & B \end{tikzcd}$$

for f a morphism $f(S) \subseteq B^*$. This define the universal property of the localization. This define the localization.

Exercise 4 p 72
Let S and T be two multiplicative subsets of A, and T^* the image of T in A_S. Then one has $A_{ST} \simeq (A_S)_{T^*}$.
Solution:
Consider the morphism $i_{ST*} : A \to (A_S)_{T^*}$, such that $i_{ST*}(ST) \subseteq T^*$ and $i_{ST*}(S) \subseteq T^*$. Then by the universal property of the localization there is a morphism $g : A_{ST} \to (A_S)T^*$ and $i : A_S \to (A_S)_{T^*}$.
Since $i(ST) \subseteq T^*$. Then by the universal property of the localization there is a morphism $h : (A_S)_{T^*} \to A_{ST}$.
By unicity of the morphism $\phi : A_{ST} \to A_{ST}$ such that $\phi \circ i_{ST} = i_{ST}$, we get that $\phi = Id_{A_{ST}} = h \circ g$. Similarly, we get $g \circ h = Id_{(A_S)_{T^*}}$. And thus the required isomorphism.

Exercise 6 p 72
Let A be an integral domain. If the localization A_S is integral over A, then $A_S = A$.
Solution:
Clearly $A \subseteq A_S$. So that it is enough to prove that $A_S \subseteq A$. Let $s \in S$ with $s \nmid b$ or $s = 1$, then there is $a_i \in A$ such that

$$(\frac{1}{s})^n + a_{n-1}(\frac{1}{s})^{n-1} + ... + a_0 = 0$$

So that

$$\frac{1 + a_{n-1}s + ... + a_0 s^n}{s^n} = 0$$

since A is integral, that is $1 + a_{n-1}b1 + ... + a_0 1^n = 0$. But then $1 = a_{n-1}s + ... + a_0 s^n$ then it implies that s is a unit. So that, $A_S = A$.

Exercise 7 p 72 (Nakayama's lemma)
Let A be a local ring with maximal ideal $\mathfrak{m}$, let M be an A-module and $N \subseteq M$ a submodule such that M/N is finitely generated. Then one has the implication:

$$M = N + \mathfrak{m}M \Rightarrow M = N$$

Solution:
Clearly, it is equivalent to prove the following: Let A be a local ring with maximal ideal $\mathfrak{m}$ and a finitely generated A-module such that $M = \mathfrak{m}$. Then $M = 0$.
For this, let $\{x_1, ..., x_n\}$ be a system of generators of M. We may suppose n minimal. There exist $\alpha_i \in \mathfrak{m}$ such that $x_n = \sum \alpha_i x_i$. Hence $(1 - \alpha_n)x_n = \sum_{i<n} \alpha_i x_i$. As $1 - \alpha_n$ is invertible, and n is assumed to be minimal, it follows that $n = 1$ and $x_n = 0$.